Direct And Inverse Theorem In Weighted Space $L_{p,w}(X)$

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Abstract— These The aim of this paper, we define the operator $V_{2n}(f)$ and use it to find the degree of best one-sided approximation of unbounded functions in weighted space $L_{p,w}(X)$ by proving direct and inverse inequalities.

Index Terms— Vallee-Poussin operator, best one-sided approximation and Ditzian-Totic modulus of smoothness.

1 INTRODUCTION

LET $X = [0,1]$, we denote by $L_\infty(X)$ [8] the set of all bounded measurable functions with usual norm $\|f\|_\infty = \sup\{|f(x)|, x \in X\}$. For ($1 \leqslant p < \infty$), let Lp(X) the set of all bounded measurable functions with norm

$$\|f\|_p = \left\{ \left(\int_0^1 |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty; (1 \le p < \infty) \right\}.$$

Further, for $\delta > 0$ the locally global norm of a function f is defined by

$$\|f\|_{\delta,p} = \left(\int_0^1 \sup\{|f(y)|^p : y \in [x - \tfrac{\delta}{2}, x + \tfrac{\delta}{2}]\} dy\right)^{\frac{1}{p}}$$

$(1 \le p < \infty)$.

Now, let W be the set of all weight functions on X. Consider $L_{p,w}(X)$ the space of all unbounded functions f on X such that $|f(x)| \le Mw(x)$,where M is positive real number, which are equipped with the following norm

$$\|f\|_{p,w} = \left(\int_0^1 \left|\frac{f(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}} < \infty.$$

For, $\delta > 0$ and $(1 \le p < \infty)$ the weighted locally global norm of $f \in Lp,w(X)$ is define by

$$\|f\|_{\delta,p,w} = \left(\int_0^1 \sup\{\left|\frac{f(y)}{w(y)}\right|^p : y \in [x - \tfrac{\delta}{2}, x + \tfrac{\delta}{2}]\} dy\right)^{\frac{1}{p}} < \infty.$$

The k^{th} locally modulus of smoothness for $f \in L_\infty$ is defined by [2]

$$\omega_k(f, x, \delta)_\infty = \sup_{|h|<\delta} \left\{ \left|\Delta_h^k f(t)\right|, t, t + kh \in [x - \tfrac{h}{2}, x + \tfrac{h}{2}] \right\}$$

where the k^{th} deference Δ_h^k is defined by

$$\Delta_h^k f(x) = \begin{Bmatrix} \sum_{i=0}^{k} (-1)^{k+i} \binom{k}{i} f\left(x + \frac{kh}{2}\right) \\ 0 \qquad\qquad \text{otherwise} \end{Bmatrix}$$

The k^{th} average modulus of smoothness for $f \in L_p(X)$ and $f \in L_{p,w}(X)$ are respectively given by

$$\tau_k(f, \delta)_p = \|\omega_k(f, ., \delta)\|_p$$

where ordinary modulus of continuity for $f \in L_p(X)$ given by

$$\omega_k(f, \delta)_p = \sup_{0<h<\delta} \left\{ \left\|\Delta_h^k f(.)\right\|_p \right\}, \delta > 0 \text{ ,and}$$

$$\tau_k(f, \delta)_{p,w} = \|\omega_k(f, ., \delta)\|_{p,w}$$

where ordinary modulus of continuity for $f \in L_{p,w}(X)$ given by

$$\omega_k(f, \delta)_{p,w} = \sup_{0<h<\delta} \left\{ \left\|\Delta_h^k f(.)\right\|_{p,w} \right\}, \delta > 0.$$

Let us define the Ditzian-Totic modulus of smoothness [8] for $f \in L_p(X)$ as $\quad w_k^\varphi(f, \delta)_p = \sup\left\|\Delta_h^{k\varphi} f(.)\right\|_p$,

Where

$$\Delta_{\varphi h}^r = \begin{Bmatrix} \sum_{i=0}^{n} (-1)^{r+i} \binom{r}{i} f(x + i\varphi h), & x + \varphi h \in X \\ 0 & \text{otherwise} \end{Bmatrix}$$

the locally Ditzian-Totic weighted modulus of smoothness for $f \in L_{p,w}(X)$ is defined by

$$w_k^\varphi(f, \delta)_{p,w} = \sup\left\|\Delta_h^{k\varphi} f(.)\right\|_{p,w}$$

where $\varphi(x) = (1-x^2)/2$.

Let w_1 and w_2 be two modulus of continuity, we say that they are equivalent if there are $k_1,k_2 > 0$ such that

$k_1 w_1(x) \le w_2(x) \le k_2 w_1(x)$, for $x > 0$.

The degree of best approximation to a given continuous function with respect to trigonometric or algebraic polynomials on interval X is given by

$$E_n(f)_\infty = \inf\{\|f - p_n\|_\infty ; p_n \in \mathbb{P}_n\}$$

where $\mathbb{P}_n$ denote the set of all trigonometric or algebraic polynomials of degree $\le$ n. While the degree of best approximation of a function $f \in L_p(X)$ with respect to trigonometric or algebraic polynomials of degree $\le n$ on X is given by

$$E_n(f)_p = \inf\{\|f - p_n\|_p ; p_n \in \mathbb{P}_n\}$$

Also, we define the degree of best weighted approximation to a given $f \in L_{p,w}(X)$ with respect to trigonometric or algebraic polynomials on X is given by

$$E_n(f)_{p,w} = \inf\{\|f - p_n\|_{p,w} ; p_n \in P_n\}$$

Now we shall define the degree of best one-sided approximation of $f \in L_p(X)$ and the degree of best one-sided weighted approximation of $f \in L_{p,w}(X)$ with respect to the trigonometric or algebraic polynomials on X are respectively given by

$$\tilde{E}_n(f)_p = \inf\{\|p_n - q_n\|_p ; p_n, q_n \in \mathbb{P}_n \text{and} q_n(x) \le f(x) \le p_n(x) \}$$

$$\tilde{E}_n(f)_{p,w} = \inf\{\|p_n - q_n\|_{p,w} ; p_n, q_n \in \mathbb{P}_n \text{and} q_n(x) \le f(x) \le p_n(x) \}$$

Weierstrass theorem tells us that $E_n \to 0$ as $n \to \infty$ for f belong to continuous functions space on X. This information can be obtained if additional information about the function f is given; for example, if we know its modulus of continuity, the class Lipα to which it belongs or the number of times it can differentiated.

In general, the smoother the function the faster $E_n(f)$ tends to zero[8].

While inverse theorem gives result in opposite direction.

The inverse results for the degree of best one-sided weighted approximation to a function $f \in L_{p,w}(X)$ with respect to algebraic polynomials implies the inverse results for the degree of best one-sided approximation to $f \in L_{p,w}(X)$ with respect to trigonometric polynomials.

Further direct-inverse theorems for algebraic polynomial of one-sided approximation in the weighted space $L_{p,w}$ are one-one correspondence with direct-inverse theorem for trigonometric polynomial of best one-sided approximation in weighed space $L_{p,w}$ to even functions.

Let $V_n(f)$ be the trigonometric polynomial operator of degree 2n-1, such that

$$V_n(f,x) = \{S_n(f,x) + S_{n+1}(f,x) + \cdots + S_{2n-1}(f,x)\} = 2\sigma_{2n}(f,x) - \sigma_n(f,x)$$

Where $S_n(f,x)$ is the Fourier series and $\sigma_n(f,x)$ is the Fejer mean see[8]. Hence $V_n(f)$ is called Vallee-poussin operator.

In (1994) Ditzian, D. Jiang and D. Leviantan [2] are obtained the equivalence for $0 < p < 1$ and $0 < \alpha < k$ between $E_n(f)_p = o(n^{-\alpha})$ and $\omega\varphi(f,t) = o(t^{\alpha})$, this result complements the know direct and inverse theorems for best approximation in space $L_p([-1,1])$, $1 \leq p < \infty$.

In (2005) A.H. AL-abdlla [1] attained the degree of approximation of 2π-periodic bounded μ-measurable functions in space $L_p(\mu)$ by proving direct and inverse inequalities of a 2π-periodic bounded μ-measurable functions and she proved that the degree approximations of these function are equivalent with degree of best one-sided approximation by trigonometric polynomials.

In (2008) Z. Esa [3] found the degree of best approximation of bounded μ-measurable function by connecting the modulus of smoothness and averaged modulus with the K-functional in $L_{p,\mu}(X)$.

2 Auxiliary Theorems and Lemmas

Theorem 2.1 [7] : *Suppose that sequence* $s_n = (a_1, a_2, \ldots, a_n)$ *converges to zero and* $F \subset Lp([-\pi,\pi])$.

Then direct inverse theorems

i. For $f \in Lp([-\pi,\pi])$, $E_n^T(f)_p = o(s_n) \Leftrightarrow f \in F$

where $E_n^T(f)_p$ denote the degree of best approximation to a function f by trigonometric polynomial of degree ≤ n.

ii. For even $f \in L_p([-\pi,\pi])$, $E_n^T(f)_p = o(s_n) \Leftrightarrow f \in F$

iii. For $f \in L_p([-1,1])$, $E_n^H(f)_p = o(s_n) \Leftrightarrow fog \in F$

where $E_n^H(f)_p$ denote the degree of best approximation to a function f by algebraic polynomial of degree ≤ n and a function g define by $g : R \to R$ such that $g(x) = \cos x$. Satisfying the implication i ⟹ ii ⟺ iii.

Lemma 2.2 *[2] : For* $f \in L_p(X)$, $(1 \leq p < \infty)$ *, we have*

$$E_n(f)_p \leq c(p)\omega_k^{\varphi}(f,\frac{1}{n})_p, \quad \varphi(x) = (1-x^2)^{\frac{1}{2}}$$

Theorem 2.3 " *Inverse theorem* " *[2] :*

For $f \in L_p(X)$ *,* $(0 < p < 1)$ *, we have*

$$\omega_k^{\varphi}(f,t)_p \leq ct^k\left(\sum_{0<h<t^{-1}}(n+1)^{kp-1}E_n(f)^p{}_p\right)^{\frac{1}{p}}.$$

Theorem 2.4 " Weierstrass Theorem " [8] :

If $f \in C[a,b]$*, then for each* $\epsilon > 0$*, there exists trigonometric polynomial T such that*

$$\|f - T\|_p < \epsilon \quad ,(1 \leq p < \infty).$$

Theorem 2.5 [7]: *For* $f \in Lp([-1,1])$*, the two modulus* $\omega_{\varphi}(f,\delta,[-1,1])_p$ *and* $\omega(fog,\delta,[-\pi,\pi])_p$ *are equivalent.*

Lemma 2.6 [6] : *Let* $f \in L_{p,w}(X)$ *,* $(1 \leq p < \infty)$*. Then*

$$E_n(f)_{p,w} \leq \tilde{E}_n(f)_{p,w} \leq cE_n(f)_{p,w}.$$

Lemma 2.7 [5]*: Let* $f \in L_p(X)$ *,* $(1 \leq p < \infty)$*. Then*

$$\tilde{E}_n(f)_p \leq c(p)\, E_n(f)_{\delta,p} \leq c\tilde{E}_n(f)_p.$$

Lemma 2.8 [6]: *Let* $f \in L_{p,w}(X)$ *,* $(1 \leq p < \infty)$*. Then*

$$\|f\|_{p,w} \leq \|f\|_{\delta,p,w} \leq c(p)\|f\|_{p,w}.$$

Lemma 2.9 [5]: *If* $p_n \in \mathbb{P}_n$ *, then*

$\|p_n\|_{p,w} = (1+\delta n)^{\frac{1}{p}}\|p_n\|_{p,w}$ *, where δ is positive real number.*

Lemma 2.10 [9]*: Let* $f \in L_{p,w}(X)$ *,* $(1 \leq p < \infty)$*. Then*

$\tilde{E}_n(f)_p \leq c_k\tau_k(f,\delta)_p$.

$$\tau_k(f,\delta)_p \leq c_k\delta^k\sum_{m=0}^{n}(m+1)^{k-1}\,\tilde{E}_n(f)_p.$$

3 Main Results

Here we shall use the operator Vn(f,x) to find the degree of best one-sided approximation in Lp,w(X) space.

Now, we need the following lemmas.

Lemma A : *Let* $f \in L_{p,w}(X)$ *,* $(1 \leq p < \infty)$*. Then*

$$\tilde{E}_n(f)_{p,w} \leq c\, E_n(f)_{\delta,p,w} \leq c\tilde{E}_n(f)_{p,w}.$$

Proof :

Consider p_n and qn are the best one-sided approximation of a function f in space $L_{p,w}(X)$ and $\tilde{E}_n(f)_{p,w} = \|p_n - q_n\|_{p,w}$,such that $q_n(x) \leq f(x) \leq p_n(x)$.

From definition of $c\,E_n(f)_{\delta,p,w}$, lemma (2.8) and let p*n be the best approximation polynomial of a function $f \in L_{p,w}(X)$

$$E_n(f)_{\delta,p,w} = \|f - p_n^*\|_{\delta,p,w} \leq c\|f - p_n^*\|_{p,w} \leq c\|p_n - q_n\|_{p,w} = c\tilde{E}_n(f)_{p,w}$$

therefore $E_n(f)_{\delta,p,w} \leq c\tilde{E}_n(f)_{p,w}$.

We shall prove the inequality $\tilde{E}_n(f)_{p,w} \leq c\,E_n(f)_{\delta,p,w}$

Since p_n^* is best approximation of f in $\mathbb{P}_n$ such that

$$\|f - p_n\|_{\delta,p,w} = E_n(f)_{\delta,p,w.}$$

Consider $p_n(x) = p_n^*(x) + \frac{n}{c}\int_X I_n(x-t)\sup\left\{\left|\frac{f(y)}{w(y)} - \frac{\varphi_n(y)}{w(y)}\right|, y \in \left[x-\frac{\delta}{2}, x-\frac{\delta}{2}\right]\right\}dt$

and $q_n(x) = p_n^*(x) - \frac{n}{c}\int_X I_n(x-t)\sup\left\{\left|\frac{f(y)}{w(y)} - \frac{\varphi_n(y)}{w(y)}\right|, y \in \left[x-\frac{\delta}{2}, x-\frac{\delta}{2}\right]\right\}dt$

clearly p_n and $q_n \in \mathbb{P}_n$, we shall to prove for every $x \in X$ the inequality $q_n(x) \leq f(x) \leq p_n(x)$ hold.

$p_n(x) = p_n^*(x) + \frac{n}{c}\int_X I_n(x-t)\sup\left\{\left|\frac{f(y)}{w(y)} - \frac{p_n^*(y)}{w(y)}\right|, y \in \left[x-\frac{\delta}{2}, x-\frac{\delta}{2}\right]\right\}dt$

$$\geq p_n^*(x) + \frac{n}{c}\int_{\frac{X}{2}} I_n(u)\sup\left\{\left|\frac{f(y)}{w(y)} - \frac{p_n^*(y)}{w(y)}\right|, y \in \left[u+t-\frac{\delta}{2}, u+t+\frac{\delta}{2}\right]\right\}du$$

$$\geq p_{n_n}^*(x) + \frac{n}{c}\left|\frac{f(x)}{w(x)} - \frac{p_n^*(x)}{w(x}\right|\frac{c}{n}$$

$$\geq p_n^*(x) + f(x) - p_n^*(x) = f(x).$$

Hence $p_n(x) \geq f(x)$.

$q_n(x) = p_n^*(x) - \frac{n}{c}\int_X I_n(x-t)\sup\left\{\left|\frac{f(y)}{w(y)} - \frac{p_n^*(y)}{w(y)}\right|, y \in \left[x-\frac{\delta}{2}, x-\frac{\delta}{2}\right]\right\}dt$

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$$\leq p_n^*(x) - \frac{n}{c}\int_{\frac{x}{2}} I_n(u)\sup\left\{\left|\frac{f(y)}{w(y)} - \frac{p_n^*(y)}{w(y)}\right|, y \in \left[x+t+\frac{\delta}{2}, x+t-\frac{\delta}{2}\right]\right\} du$$

$$\leq p_n^*(x) - \frac{n}{c}\left|\frac{f(x)}{w(x)} - \frac{p_n^*(x)}{w(x}\right|\frac{c}{n} \leq p_n^*(x) + f(x) - p_n^*(x) = f(x)$$

so, $q_n(x) \leq f(x)$

Thus

$$\|p_n - q_n\|_{p,w} = 2\left(\int_X \left(\frac{n}{c}\int_X I_n(x-t)\sup\left\{\left|\frac{f(y)}{w(y)} - \frac{p_n^*(y)}{w(y)}\right|, y \in \left[x-\frac{\delta}{2}, x-\frac{\delta}{2}\right]\right\} dt\, dy\right)^{\frac{1}{p}}$$

$$\leq \frac{2n}{c}\left(\int_X \left(\frac{c}{n}\int_{\frac{x}{2}} I_n(u)\sup\left\{\left|\frac{f(y)}{w(y)} - \frac{p_n^*(y)}{w(y)}\right|, y \in \left[u+t-\frac{\delta}{2}, u+t+\frac{\delta}{2}\right]\right\} du\, dy\right)^{\frac{1}{p}}$$

$$= 2\|I_n\|\left(\int_X \sup\left\{\left|\frac{f(y)}{w(y)} - \frac{p_n^*(y)}{w(y)}\right|^p y \in \left[u+t-\frac{\delta}{2}, u+t+\frac{\delta}{2}\right]\right\} dy\right)^{\frac{1}{p}} = cE_n(f)_{\delta,p,w}$$

we get $\tilde{E}_n(f)_{p,w} \leq c\, E_n(f)_{\delta,p,w}$.

Lemma B : *Let* $f \in L_{p,w}(X)$ *, (* $1 \leq p < \infty$ *). Then*

$$\tilde{E}_n(f)_{p,w} \leq c_k \tau_k(f,\delta)_{p,w}$$

Proof : We have

$$\tilde{E}_n(f)_{p,w} = \inf\{\|p_n - q_n\|_{p,w} ; p_n, q_n \in P_n \text{and} q_n(x) \leq f(x) \leq p_n(x)\}$$

$$= \inf\left\{\left(\int_X \left|\frac{p_n(x)}{w(x)} - \frac{q_n(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}}\right\} = \tilde{E}_n\left(\frac{f}{w}\right)_p$$

Since $\frac{f}{w}$ is integrable function, then by using lemma (2.9) (i), we get

$\tilde{E}_n(\frac{f}{w})_p \leq c_k \tau_k(\frac{f}{w},\delta)_p = c_k \tau_k(f,\delta)_{p,w}$.

Lemma C : *Let* $f \in L_{p,w}(X)$ *, for (* $1 \leq p < \infty$ *). Then*

$$\tau_k(f,\delta)_{p,w} \leq c_k \delta^k \textstyle\sum_{m=0}^{n}(m+1)^{k-1}\, \tilde{E}_m(f)_{p,w}$$

Proof : We have

$$\tau_k(f,\delta)_{p,w} = \|\omega_k(f,.,\delta)\|_{p,w} \leq 2^p\|\omega_k(f,.,\delta)\|_{p,w}$$

$$= 2^p\left\|\omega_k(\frac{f}{w},.,\delta)\right\|_p = 2^p \tau_k(\frac{f}{w},\delta)_p$$

By using lemma (2.9) (ii)

$$\leq c_k \delta^k \sum_{m=0}^{n}(m+1)^{k-1}\, \tilde{E}_m(\frac{f}{w})_p$$

Since $\tilde{E}_n(\frac{f}{w})_p = \inf\left\{\left\|\frac{p_n}{w} - \frac{q_n}{w}\right\|_p\right\} = \inf\{\|p_n - q_n\|_{p,w}\} = \tilde{E}_n(f)_{p,w}$

We get $\tau_k(f,\delta)_{p,w} \leq c_k \delta^k \sum_{m=0}^{n}(m+1)^{k-1}\, \tilde{E}_m(f)_{p,w}$

Lemma D: *For* $f \in L_{p,w}(X)$ *, we have that the two modulus* $\omega(fog,\delta,)_{p,w}$ *and* $\omega_\varphi(f,\delta,)_{p,w}$ *are equivalent.*

Proof : We have

$$\omega_\varphi(f,\delta,)_{p,w} = \sup_{0<h<\delta}\|\Delta_{\varphi h}f(.)\|_{p,w}$$

$$= \sup_{0<h<\delta}\left\{\left(\int_0^1 \left|\frac{f(x+\varphi(x)h) - f(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}}\right\}$$

$$= \sup_{0<h<\delta}\left\{\left(\int_0^1 \left|\frac{f(x+\varphi(x)h}{w(x)} - \frac{f(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}}\right\} = \sup_{0<h<\delta}\left\|\Delta_{\varphi h}\frac{f(.)}{w(.)}\right\|_p$$

$$= w_\varphi\left(\frac{f}{w},\delta,\right)_p .$$

From the definition of $f \in L_{p,w}(X)$ we get $\frac{f}{w}$ integerable , also we can show that $\frac{fog}{w}$ integreable function.

Similarly, we can easily to show that

$\omega(fog,\delta,)_{p,w} = \omega(\frac{fog}{w},\delta,)_p$

From theorem (2.5), we get $\omega_\varphi(f,\delta,)_{p,w}$ and $\omega(fog,\delta,)_{p,w}$ are equivalent.

Lemma E: *For* $f \in L_{p,w}(X)$*, we have the following are equivalent :*

i. $E_n^H(f)_{p,w} = o(\frac{1}{n})$.

ii. $E_n^T(f)_{p,w} = o(\frac{1}{n})$.

iii. $\omega_\varphi(f,\frac{1}{n})_{p,w} = o(\frac{1}{n})$.

iv. $\omega(fog,\frac{1}{n})_{p,w} = o(\frac{1}{n})$.

v. There exist a constant c and the set of polynomials Pn of degree $\leq n$ *satisfying*

$\|f - p_n\|_{p,w} = o(\frac{1}{n})$.

Proof :

Let $E_n^H(f)_{p,w} = o(\frac{1}{n})$ by using theorem (2.1), we can easily show that $E_n^T(f)_{p,w} = o(\frac{1}{n})$ and the converse is true, therefore i $\Leftrightarrow$ ii.

By using lemma (2.2) and theorem (2.3) we obtain $E_n^T(f)_{p,w}$ is equivalent to $\omega_\varphi(f,\frac{1}{n},)_{p,w}$, hence ii $\Leftrightarrow$ iii.

From lemma (D), we have $\omega_\varphi(f,\frac{1}{n},)_{p,w}$ and $\omega(fog,\frac{1}{n})_{p,w}$ are equivalent, we get iii $\Leftrightarrow$ iv.

Finally, we have $\omega(fog,\frac{1}{n})_{p,w} = o(\frac{1}{n})$, from theorem (2.4) and $\frac{f}{w}$ is continuous function then $\left\|\frac{f}{w} - \frac{p_n}{w}\right\|_p = \|f - p_n\|_{p,w} < \epsilon$

$$\Leftrightarrow \|f - p_n\|_{p,w} < c\omega(f,\frac{1}{n})_{p,w}$$

$$\Leftrightarrow \|f - p_n\|_{p,w} = o(\frac{1}{n})$$

So iv $\Leftrightarrow$ v.

Remark :

For $f \in L_{p,w}(X)$ and by using theorem (2.1) we can obtain easily that direct-inverse theorems for algebraic polynomial of one-sided approximation are 1-1 correspondence with direct-inverse theorems for trigonometric polynomial of best one-sided approximation to even function.

Theorem 3.1 " **Direct Theorem** " :

Let $f \in L_{p,w}(X)$ *, for (* $1 \leq p < \infty$ *). Then*

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$$\|f - V_n(f)\|_{p,w} \le c_k \tau_k(f,\delta)_{p,w}$$

Proof : $\|f - V_n(f)\|_{p,w} = \|f - p_n + p_n - V_n(f)\|_{p,w}$

$$\le \|f - p_n\|_{p,w} + \|p_n - V_n(f)\|_{p,w}$$

Since V_n preserver of trigonometric polynomial and by using linearity we get

$$= \|f - p_n\|_{p,w} + \|V_n(p_n) - V_n(f)\|_{p,w}$$
$$= \|f - p_n\|_{p,w} + \|V_n(p_n - f)\|_{p,w}$$
$$= \|f - p_n\|_{p,w} + \|p_n - f\|_{p,w}$$
$$= 2\|f - p_n\|_{p,w}$$

Let p_n and q_n be the trigonometric polynomials of degree less than or equal n best one-sided approximation of a function $f \in L_{p,w}(X)$ such that

$$q_n(x) \le f(x) \le p_n(x)\ , x \in X.$$

So, $\|f - V_n(f)\|_{p,w} \le c_1\|p_n - q_n\|_{p,w} \le c_2\, \tilde{E}_n(f)_{p,w}$

By using Lemma (2.10) (i), we get

$\|f - V_n(f)\|_{p,w} \le c_k \tau_k(f,\delta)_{p,w}$.

We can prove the direct Theorem by standard method from the following lemma.

Lemma 3.2: *If $f \in L_{p,w}(X)$ and $\delta > 0$, then*

$$E_n^H(f)_{p,w} \le w(fog,\delta,)_{p,w} \le k\, w_\varphi(f,\delta,)_{p,w} \le k\, w(f,\delta,)_{p,w}\,.$$

Proof : By using lemma (2.2), then $E_n(f)_{p,w} \le k\omega(f,\delta)_{p,w}$,k is constant and from lemma (D), we get $E_n(f)_{p,w} \le k\omega(fog,\delta)_{p,w}$

Also by using lemma (D) and definition of w we have

$$\omega(fog,\delta)_{p,w} \le k\omega_\varphi(f,\delta)_{p,w} \le k\omega(f,\delta)_{p,w}$$

We obtain

$$E_n^H(f)_{p,w} \le \omega(fog,\delta,)_{p,w} \le k\,\omega_\varphi(f,\delta,)_{p,w} \le k\,\omega(f,\delta,)_{p,w}\,.$$

Theorem 3.4 " Inverse Theorem "

Let $f \in L_{p,w}(X)$, for ($1 \le p < \infty$) , then

$$\tau_k(f,\delta)_{p,w} \le c_k \delta^k \sum_{m=0}^{n} (m+1)^{k-1} \|f - V_n(f)\|_{p,w}$$

Proof :

Consider $E_n(f)_{p,w} = \inf\|f - V_n(f)\|_{p,w}$

From lemma (C)

$\tau_k(f,\delta)_{p,w} \le c_k \delta^k \sum_{m=0}^{n} (m+1)^{k-1}\, \tilde{E}_m(f)_{p,w}$

Also by using lemma (A) we get

$$\le c_k \delta^k \sum_{m=0}^{n} (m+1)^{k-1} E_m(f)_{p,w}$$

$$\le c_k \delta^k \sum_{m=0}^{n} (m+1)^{k-1} \|f - V_n(f)\|_{p,w}$$

REFERENCES

[1] H. AL-Abdulla,"On equi approximation of bounded μ-measurable functions in $L_P(\mu)$-space", Thesis presented to college of Education, university of Baghdad, Iraq, (2005).

[2] Z. Ditzian , D. Jiong and D. Leviatan ,"Inverse theorem for best polynomial approximation in $L_P(1 < p < 0)$ ", Proceeding of American Math. society Vol. 120 No.1 (1994).

[3] Z. Esa , "integral average modulus with application in the approximation of function in the $L_{p,\mu}$ spaces ", Thesis presented to college of education, Al-mustansirya university, Iraq, (2008).

[4] H. Fadhil , "On the constraint spline approximation" , Thesis presented to college of Education, university of Babylon, Iraq (2010).

[5] V. H. Hritov , "Best one-sided approximation and mean approximation by interpolation polynomials of periodic functions" ,Math. Balkanica, New series, (1989),418-429.

[6] S. K. Jassim , "one-sided approximation with discrete operator", ph.D Thesis, Sofia, Bulgaria (1991).

[7] S.K. Jassim and E.S. Bhaya , "Direct and Inverse theorem for best multi-approximation in Lp ($1 \le p < \infty$), Journal of Math. and physics Vol. 17, No.3, Iraq, (2002).

[8] G. G. Lornentz ,"Approximation of function", U.S.A. (1966).

[9] V. A. Popov and L. Alekksandrov, "one-sided trigonometrical approximation of periodic multi-variant functions ", Math. Balkanica, (1987), 3-4.